

EHRHART ANALOGUE OF THE h -VECTOR.

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ABSTRACT. We consider a formula of Stanley that expresses the Ehrhart generating polynomial of a polyhedral complex in terms of the h -polynomials of toric varieties. We prove that the coefficients in this expression are all non-negative and show that these coefficients can be found using the decomposition theorem in intersection cohomology.

1. INTRODUCTION

The title of this note comes from Example 7.13 in Stanley's paper [8]. In that example Stanley considers the Ehrhart problem of counting lattice points in a polyhedral complex C . In analogy with decomposing the h -vector of a subdivision of C into "local h -vectors", he decomposes the Ehrhart generating polynomial of C into the same local h -vectors with coefficients $c_{\sigma,i}$ for $\sigma \in C$, $i \geq 0$. We prove Conjecture 7.14 in [8] that these coefficients $c_{\sigma,i}$ are non-negative by interpreting them in terms of orbifold cohomology.

Even though Stanley's definition of the numbers $c_{\sigma,i}$ is combinatorial, it is more illuminating to define them using orbifold cohomology. It is well-known that counting lattice points in a polyhedral complex is equivalent to studying orbifold cohomology of some toric orbifold, a relationship that is analogous to the equivalence between counting faces of a (simplicial) polyhedral complex and ordinary cohomology. In relating the combinatorics and cohomology we follow the article of Mustata and Payne [7] where this equivalence is used to study a conjecture of Hibi.

Let us start by recalling the definition of orbifold cohomology defined by Chen and Ruan [6]. Later we will specialize to the case of toric varieties studied by Borisov, Chen and Smith [4]. We are only interested in the dimensions of the orbifold cohomology spaces, not the ring structure. These dimensions were computed in a more general setting by Batyrev [2] and Batyrev-Dais [3].

Given a complete Gorenstein orbifold X , one decomposes its inertia scheme:

$$I(X) = \coprod_i X_i$$

and defines the orbifold cohomology of X :

$$H_{orb}^k(X; \mathbb{R}) = \oplus_i H^{k-2s_i}(X_i; \mathbb{R}),$$

where $s_i \geq 0$ is the "age" of the component X_i . If $Y \rightarrow X$ is a crepant resolution of X , then the orbifold cohomology of X is isomorphic to the ordinary cohomology of Y .

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When X_Δ is a toric orbifold defined by a complete simplicial Gorenstein fan Δ , then $I(X_\Delta)$ is a union of orbit closures V_σ for $\sigma \in \Delta$ and we have

$$(1.1) \quad H_{orb}^k(X_\Delta; \mathbb{R}) = \bigoplus_{\sigma \in \Delta} \bigoplus_{i \geq 0} [H^{k-2i}(V_\sigma; \mathbb{R})]^{c_{\sigma,i}}.$$

The coefficients $c_{\sigma,i}$ have a combinatorial meaning: they count lattice points in the interior of $\text{Box}(\sigma)$ (see Section 3 below for the definition of $\text{Box}(\sigma)$).

Let now X be an arbitrary complete Gorenstein variety that admits a crepant resolution $Y \rightarrow X$, where Y is an orbifold. Define the *stringy cohomology* of X by

$$H_{str}^k(X; \mathbb{R}) := H_{orb}^k(Y; \mathbb{R}).$$

By a result of Yasuda [9] this does not depend on the choice of Y . The dimensions of the stringy cohomology spaces $H_{str}^k(X; \mathbb{R})$ are given by Batyrev's stringy Betti numbers [2]. An arbitrary Gorenstein variety may not have any crepant orbifold resolution, however, a Gorenstein toric variety X_Δ always has one, hence the stringy cohomology $H_{str}^k(X_\Delta; \mathbb{R})$ is well-defined.

Note that orbifold cohomology was defined using the decomposition of the inertia scheme. The behavior of the cohomology under crepant morphisms was a theorem. For stringy cohomology we reverse the situation: we define it using a crepant orbifold resolution. Its decomposition then becomes a theorem.

Theorem 1.1. *Let X_Δ be a complete Gorenstein toric variety. There exist non-negative integers $c_{\sigma,i}$ such that*

$$H_{str}^k(X_\Delta; \mathbb{R}) = \bigoplus_{\sigma \in \Delta} \bigoplus_{i \geq 0} [IH^{k-2i}(V_\sigma; \mathbb{R})]^{c_{\sigma,i}}, \quad k \geq 0,$$

where $IH^*(V_\sigma)$ is the intersection cohomology of V_σ . The numbers $c_{\sigma,i}$ depend on the cone σ only, not on the fan Δ .

The numbers $c_{\sigma,i}$ are the ones calculated by Stanley in [8]. The theorem is proved by applying the decomposition theorem in intersection cohomology to the crepant orbifold resolution $Y \rightarrow X_\Delta$.

Let us now explain the relation with lattice point counting. If P is a d -dimensional lattice polytope in \mathbb{R}^n , its Ehrhart generating function is

$$F_P(t) = \sum_{j=0}^{\infty} \#(jP \cap \mathbb{Z}^n) t^j = \frac{\delta_0 + \delta_1 t + \dots + \delta_d t^d}{(1-t)^{d+1}}$$

for some non-negative integers $\delta_0, \dots, \delta_d$. Similarly, when C is a complex of lattice polytopes in \mathbb{R}^n , such that C is topologically a d -sphere, then

$$(1.2) \quad F_C(t) = \sum_{j=0}^{\infty} \#(jC \cap \mathbb{Z}^n) t^j = \frac{\delta_0 + \delta_1 t + \dots + \delta_{d+1} t^{d+1}}{(1-t)^{d+1}},$$

with $\delta_0, \dots, \delta_{d+1}$ non-negative integers. Denote by $\delta_C(t)$ the polynomial in the numerator of the fraction in (1.2).

We will consider the case where the complex C is defined by a complete Gorenstein fan Δ in \mathbb{R}^n . Recall that Δ being Gorenstein means that there exists a conewise

linear integral function $K_\Delta : \mathbb{R}^n \rightarrow \mathbb{R}$, such that $K_\Delta(v) = 1$ for all primitive generators v of the rays of Δ . Let $C = K_\Delta^{-1}(1)$. Then in the simplicial case the numbers δ_k in Equation (1.2) are (see [4, 7])

$$\delta_k = \dim H_{orb}^{2k}(X_\Delta; \mathbb{R}).$$

Since stringy cohomology was defined by a crepant resolution, for a general X_Δ we have

$$\delta_k = \dim H_{str}^{2k}(X_\Delta; \mathbb{R}).$$

For a toric variety X write its h -polynomial:

$$h_X(t) = \sum_{k \geq 0} \dim IH^{2k}(X; \mathbb{R}) t^k.$$

Also let $c_\sigma(t)$ be the polynomial

$$c_\sigma(t) = \sum_{j \geq 0} c_{\sigma,j} t^j,$$

with $c_{\sigma,j}$ defined in Theorem 1.1. Then Theorem 1.1 implies by computing dimensions on both sides:

Corollary 1.2.

$$\delta_C(t) = \sum_{\sigma \in \Delta} c_\sigma(t) h_{V_\sigma}(t).$$

Note that in the corollary all polynomials have non-negative coefficients. The polynomial $c_\sigma(t)$ depends on the cone σ only, while the polynomial $h_{V_\sigma}(t)$ depends on the poset of $\text{Star}_\Delta(\sigma)$ only.

To decompose the stringy cohomology as a direct sum of intersection cohomologies, we use the combinatorial theory of locally free and flabby sheaves on the fan Δ ([1, 5]). We construct a locally free and flabby sheaf \mathcal{E}_Δ , called the Ehrhart sheaf, such that the global sections of \mathcal{E}_Δ give the equivariant stringy cohomology of X_Δ . Applying the combinatorial decomposition theorem to \mathcal{E}_Δ then proves Theorem 1.1.

Let us mention a few generalizations of Theorem 1.1. The construction of the Ehrhart sheaf \mathcal{E}_Δ and the decomposition of this sheaf makes sense for any (rational) fan. Theorem 1.1 and Corollary 1.2 hold, for example, if the fan Δ is quasi-convex [1]. With the same tools one can even treat the case of abstract fans corresponding to abstract polyhedral complexes, but we will not pursue these generalizations here. Let us only bring the analogue of Corollary 1.2 in the following quasi-convex case.

Suppose the complex C consists of a single lattice polytope P . Then the fan Δ consists of a single cone over the polytope P and all its faces. The h -polynomial of the affine toric variety X_Δ is usually called the g -polynomial of the polar dual polytope P° . Similarly, the h -polynomial of V_σ for a cone σ corresponding to a face $F \leq P$ is the g -polynomial of the dual face $F^* \leq P^\circ$. The formula in Corollary 1.2 can now be written as:

$$\delta_P(t) = \sum_{F \leq P} c_F(t) g_{F^*}(t),$$

where we replaced $c_\sigma(t)$ by the corresponding $c_F(t)$. Again, all polynomials in this formula have non-negative integer coefficients. The polynomial $c_F(t)$ depends on the face F only, the polynomial $g_{F^*}(t)$ depends on the poset of the dual face F^* only.

2. COMBINATORIAL DECOMPOSITION THEOREM

We recall briefly the decomposition theorem for locally free and flabby sheaves on a fan Δ ([1, 5]). All vector spaces are over the field \mathbb{R} .

Let Δ be a polyhedral fan in \mathbb{R}^n . The fan (as a finite set of cones) is given the topology in which open sets are subfans of Δ . A sheaf of vector spaces F in this topology consists of the data:

- A vector space F_σ for each $\sigma \in \Delta$, the stalk of F at σ .
- A linear map $res_\tau^\sigma : F_\sigma \rightarrow F_\tau$ for τ a face of σ , such that $res_\sigma^\sigma = Id$ and $res_\rho^\tau \circ res_\tau^\sigma = res_\rho^\sigma$ whenever $\sigma > \tau > \rho$. These maps are called restriction maps.

To a sheaf F on Δ we can apply the usual constructions, such as taking global sections or computing the sheaf cohomology. Given a subdivision of fans $\phi : \hat{\Delta} \rightarrow \Delta$ and a sheaf F on $\hat{\Delta}$, we let $\phi_*(F)$ be the push-forward of F .

Recall that a sheaf F is flabby if for any open sets $U \subset V$, sections on U can be lifted to sections on V . In the case of fans this amounts to the surjectivity of the maps

$$F_\sigma \rightarrow \Gamma(F, \partial\sigma)$$

for all cones $\sigma \in \Delta$.

Let $\mathcal{A} = \mathcal{A}_\Delta$ be the sheaf defined by:

- \mathcal{A}_σ is the space of polynomial functions on σ .
- res_τ^σ is the restriction of functions.

The sheaf \mathcal{A} is a graded sheaf of rings. It is flabby if and only if the fan Δ is simplicial. If the fan is complete and simplicial, then global sections of \mathcal{A} form the T -equivariant cohomology ring of X_Δ , where T is the torus.

Let \mathcal{F} be a sheaf of \mathcal{A} -modules. \mathcal{F} is called locally free if \mathcal{F}_σ is a finitely generated free \mathcal{A}_σ -module for all $\sigma \in \Delta$. The decomposition theorem states that a locally free flabby sheaf \mathcal{F} can be decomposed as a direct sum of elementary sheaves:

$$\mathcal{F} = \bigoplus_{\sigma \in \Delta} \bigoplus_i \mathcal{L}_\sigma^{c_{\sigma,i}}[i].$$

The sheaves \mathcal{L}_σ are indecomposable locally free flabby sheaves supported on $\text{Star } \sigma = \{\pi \geq \sigma\}$. In the simplicial case \mathcal{L}_σ is simply the sheaf \mathcal{A} restricted to $\text{Star } \sigma$. The non-negative integers $c_{\sigma,i}$ depend on the restriction of \mathcal{F} to the cone σ and all its faces only.

Assume now that Δ is complete. Then the global sections of the sheaf \mathcal{L}_σ form the T -equivariant intersection cohomology of the orbit closure V_σ . The equivariant cohomology is related to the non-equivariant cohomology as follows. Let A be the ring of global polynomial functions on \mathbb{R}^n . Then \mathcal{L}_σ is a sheaf of A -modules. The space of global sections of \mathcal{L}_σ forms a free A -module and we have

$$\Gamma(\mathcal{L}_\sigma, \Delta) = IH_T^*(V_\sigma; \mathbb{R}) \simeq IH^*(V_\sigma; \mathbb{R}) \otimes_{\mathbb{R}} A.$$

The degree convention is that a section of degree k gives a cohomology class of degree $2k$. In particular, the Hilbert function of the space of global sections of L_σ is

$$\sum_{j=0}^{\infty} \dim \Gamma(\mathcal{L}_\sigma, \Delta)_j t^j = \frac{h_0 + h_1 t + \dots + h_n t^n}{(1-t)^n},$$

where $h_k = \dim IH^{2k}(V_\sigma; \mathbb{R})$.

3. GORENSTEIN FANS

Let σ be a rational polyhedral pointed cone in \mathbb{R}^n . Let v_1, \dots, v_m be the primitive lattice points on the edges of σ . Recall that σ is called Gorenstein if there exists a linear function $K_\sigma : \sigma \rightarrow \mathbb{R}$ taking integral values on $\mathbb{Z}^n \cap \sigma$ and such that $K_\sigma(v_i) = 1$ for $i = 1, \dots, m$. A fan Δ is Gorenstein if all cones $\sigma \in \Delta$ are Gorenstein. On a Gorenstein fan the functions K_σ glue to a continuous conewise linear function K_Δ .

Consider a Gorenstein d -dimensional cone σ and the power series

$$F_\sigma(t) = \sum_{j=0}^{\infty} \#(K_\sigma^{-1}(j) \cap \mathbb{Z}^n) t^j.$$

Since this is the Ehrhart generating function for the polytope $P = K_\sigma^{-1}(1)$, we can write it as a rational function

$$F_\sigma(t) = \frac{\delta_0 + \delta_1 t + \dots + \delta_{d-1} t^{d-1}}{(1-t)^d}.$$

Note that if \mathcal{E}_σ is the free graded \mathcal{A}_σ -module

$$\mathcal{E}_\sigma = \mathcal{A}_\sigma^{\delta_0} \oplus \mathcal{A}_\sigma^{\delta_1}[-1] \oplus \dots \oplus \mathcal{A}_\sigma^{\delta_{d-1}}[-(d-1)],$$

then its Hilbert series is precisely $F_\sigma(t)$. We construct a locally free and flabby sheaf \mathcal{E} on Δ with stalks \mathcal{E}_σ as above and call it the Ehrhart sheaf. By the decomposition theorem there is up to an isomorphism a unique locally free flabby sheaf on Δ with the given stalks, hence the Ehrhart sheaf is unique.

Let us start with the case where Δ is simplicial. Let σ be a simplicial cone with primitive generators v_1, \dots, v_d , and let $\text{Box}(\sigma)$ be

$$\text{Box}(\sigma) = \{v \in \mathbb{Z}^n \mid v = \alpha_1 v_1 + \dots + \alpha_d v_d \text{ for some } 0 \leq \alpha_i < 1\}.$$

Define \mathcal{E}_σ to be the free graded \mathcal{A}_σ -module with basis $\text{Box}(\sigma)$, where a basis element $v \in \text{Box}(\sigma)$ has degree $K_\sigma(v)$. (This definition of \mathcal{E}_σ agrees with the one given above.) If τ is a face of σ then $\text{Box}(\tau) \subset \text{Box}(\sigma)$, and we get the restriction map $\mathcal{E}_\sigma \rightarrow \mathcal{E}_\tau$ by sending a basis element $v \in \text{Box}(\sigma)$ to v if $v \in \text{Box}(\tau)$ and to zero otherwise. This defines a locally free sheaf $\mathcal{E} = \mathcal{E}_\Delta$ on Δ .

Lemma 3.1. *The sheaf \mathcal{E}_Δ is flabby on the simplicial fan Δ .*

Proof. Note that \mathcal{E}_Δ decomposes into a finite direct sum

$$\mathcal{E}_\Delta = \bigoplus_{v \in \mathbb{Z}^n} \mathcal{E}_v,$$

where

$$\mathcal{E}_{v,\sigma} = \begin{cases} \mathcal{A}_\sigma & \text{if } v \in \text{Box}(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

If σ is the smallest cone such that $v \in \text{Box}(\sigma)$, then

$$\mathcal{E}_v = \mathcal{A}|_{\text{Star } \sigma} = \mathcal{L}_\sigma,$$

which is flabby. \square

We note that the decomposition $\mathcal{E} = \oplus_v \mathcal{E}_v$ in the proof above corresponds to the decomposition (1.1) of the orbifold cohomology of X_Δ .

Let $\Gamma(\mathcal{E}_\Delta, \Delta)_k$ be the degree k component of the graded vector space of global sections.

Lemma 3.2. *We have*

$$\dim \Gamma(\mathcal{E}_\Delta, \Delta)_k = \#(K_\Delta^{-1}(k) \cap \mathbb{Z}^n).$$

Proof. Define a sheaf F of graded vector spaces on Δ as follows. The stalk F_σ has basis $\sigma \cap \mathbb{Z}^n$ with $v \in \sigma \cap \mathbb{Z}^n$ having degree $K_\sigma(v)$. The restriction map res_τ^σ sends a basis element v to the same basis element or to zero as appropriate. Clearly F is a flabby sheaf of vector spaces on Δ satisfying the equality stated for \mathcal{E}_Δ . Similarly to the case of locally free flabby \mathcal{A} -modules, there is up to an isomorphism a unique graded flabby sheaf of vector spaces on Δ with the given stalks. Hence it suffices to prove that \mathcal{E}_Δ and F have the same stalks. This is clear. \square

Let now Δ be an arbitrary rational polyhedral complete Gorenstein fan in \mathbb{R}^n . Then Δ has a simplicial subdivision

$$\phi : \hat{\Delta} \rightarrow \Delta,$$

such that $K_{\hat{\Delta}} = K_\Delta \circ \phi$. Such a subdivision ϕ is called crepant. Define

$$\mathcal{E}_\Delta = \phi_* \mathcal{E}_{\hat{\Delta}}.$$

Proposition 3.3. *\mathcal{E}_Δ is a locally free flabby sheaf on Δ , such that*

$$\dim \Gamma(\mathcal{E}_\Delta, \Delta)_k = \#(K_\Delta^{-1}(k) \cap \mathbb{Z}^n).$$

Proof. \mathcal{E}_Δ is a locally free and flabby by [1, 5]. The second statement follows from the equality

$$\Gamma(\mathcal{E}_\Delta, \Delta)_k = \Gamma(\mathcal{E}_{\hat{\Delta}}, \hat{\Delta})_k$$

and the previous lemma. \square

As explained in the introduction, the proposition states that $\Gamma(\mathcal{E}_\Delta, \Delta)$ gives the equivariant stringy cohomology of X_Δ . Since \mathcal{E}_Δ is locally free and flabby, we decompose it as

$$\mathcal{E}_\Delta = \bigoplus_{\sigma \in \Delta} \bigoplus_i \mathcal{L}_{\sigma, i}^{c_{\sigma, i}}[i],$$

for some non-negative integers $c_{\sigma, i}$ that depend on the cone σ only. Taking global sections of both sides proves Theorem 1.1.

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